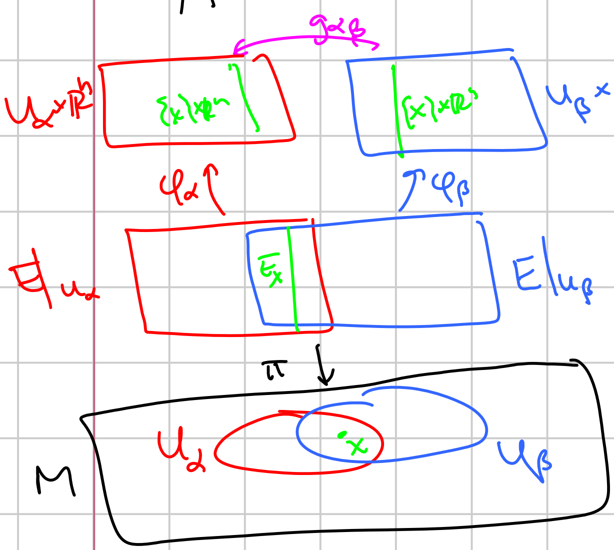
Math 612 part 5 — Thom isomorphism

Vector bundles

A fiber bundle with fiber $\mathbb{R}^n$ is a <u>vector bundle</u> if transition functions are in $GL(n,\mathbb{R}) \subset \mathrm{Diff}(\mathbb{R}^n)$:

$$\begin{array}{c} E \\ \pi \downarrow \\ M^m \end{array} \qquad \text{open cover } \{U_\alpha\} \text{ of } M, \text{ maps } \varphi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\sim"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & U_\alpha \times \mathbb{R}^n \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$



$x \in U_\alpha \cap U_\beta \leadsto g_{\alpha\beta}(x) = \varphi_\alpha \circ \varphi_\beta^{-1}|_{\{x\}\times\mathbb{R}^n} \in GL(n,\mathbb{R}) \subset \mathrm{Diff}(\mathbb{R}^n)$.

$\{(U_\alpha, \varphi_\alpha)\}$ are a <u>trivialization</u> of E.

<u>Def</u> $\begin{array}{c} E \\ \downarrow \\ M \end{array}, \begin{array}{c} E' \\ \downarrow \\ M \end{array}$ are <u>isomorphic</u> as vector bundles if $\exists\, \psi : E \to E'$

with
$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & E' \arrow[dl] \\ & M & \end{tikzcd}$$
and $\psi|_{E_x} : E_x \to E'_x$ is in $GL(n,\mathbb{R})$.

(or <u>bundle map</u> if $\psi|_{E_x}$ is linear).

One way to describe a trivialization: frames.

<u>Def</u> A <u>section</u> of E over $U \subset M$ is $s: U \to E|_U$ s.t. $\pi \circ s = id$.

A <u>frame</u> for E over U is $(s_1, \dots, s_n)$ s.t. $s_1(x), \dots, s_n(x)$ form a basis for $E|_x$ $\forall x \in U$.

Note: trivialization $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n \iff$ frame for E over U_α:

$\Rightarrow$: $s_i(x) = \varphi_\alpha^{-1}(x)(\vec{e}_i)$.

$\Leftarrow$: $\varphi_\alpha\left(f_1(x) s_1(x) + \dots + f_n(x) s_n(x)\right) = (f_1(x), \dots, f_n(x))$.

Given two frames near a point, one is a linear comb of the other $\Rightarrow$ transition fns.

<u>Def</u> $H \subset GL(n, \mathbb{R})$ subgp. The <u>structure group of E can be reduced to H</u> if $\exists$ trivialization st. $g_{\alpha\beta}(x) \in H$.

<u>Prop</u> The structure group of any vector bundle can be reduced to $O(n)$.

<u>Pf</u> Place a Riemannian metric on the fibers of E:
Smoothly varying metric (pos def bilinear form) on E_x.

Any frame on $E|_{U_\alpha}$ can be made orthonormal (and still smooth) by Gram-Schmidt. Then on overlaps,
$(s_1(x), \dots, s_n(x))$, $(s_1'(x), \dots, s_n'(x))$ are related by an orthogonal matrix $\Rightarrow$ with respect to this trivialization, $g_{\alpha\beta}(x) \in O(n)$. $\square$

Def E is orientable as a vector bundle if the structure group can be reduced to $GL^+(n,\mathbb{R})$ or equivalently $SO(n)$.

(note: Gram-Schmidt preserve the sign of the determinant)

Ex $E = \begin{array}{c} TM \\ \downarrow \\ M \end{array}$ tangent bundle. Transition functions for TM are given by the Jacobian of transition fns for M, so M is orientable $\Leftrightarrow$ TM is orientable as a vector bundle.

CV Cohomology

$\mathbb{R}^n \to \begin{array}{c} E \\ \downarrow \\ M^m \end{array}$ vector bundle. On E, we can consider

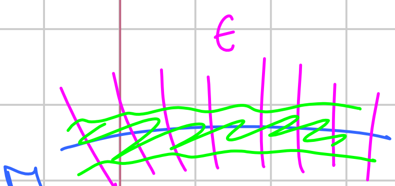
$$H^*(E) \cong H^*(M) \quad \text{Poincaré (homotopy invariance)}$$

or

$$H^k_c(E) \cong H^{*-n}_c(M) \quad \text{if } E, M \text{ are orientable}$$

$$H^k_c(E) \cong \left(H^{m+n-k}(E)\right)^* \cong \left(H^{m+n-k}(M)\right)^* \cong H^{k-n}_c(M).$$

Or we can consider a third cohomology.



Def $\Omega^*_{cv}(E) = \{\omega \in \Omega^*(E) \mid \forall K \subset M \text{ cpt}, \ \operatorname{supp}\omega \cap \pi^{-1}(K) \text{ is compact}\}$.

forms with compact vertical support.

$\hookrightarrow$ note in particular $\forall x \in M$, $\omega|_{E_x} \in \Omega^*_c(E_x)$

Note $\Omega^*_c(E) \subset \Omega^*_{cv}(E) \subset \Omega^*(E)$

and $d : \Omega^*_{cv}(E) \to \Omega^*_{cv}(E)$. **Def** $\underline{H^*_{cv}(E) = \ker d / \operatorname{im} d}$.

Thom Isomorphism Thm $\begin{smallmatrix}E\\ \downarrow\\ M\end{smallmatrix}$ rank n vector bundle, $\mathcal{U}$ = good cover of M, $\mathcal{H}^*_{cv}$ = presheaf on M defined below. Then

$$H^*_{cv}(E) \cong \check{H}^{*-n}(\mathcal{U}, \mathcal{H}^n_{cv}).$$

If E is orientable as a vector bundle, then

$$H^*_{cv}(E) \cong H^{*-n}(M).$$

Ex if $M = pt$ then $H^*_{cv}(E) \cong H^*_c(\mathbb{R}^n) \cong H^{*-n}(pt)$.

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How do we prove this?

$\mathcal{U}$ = good cover of M, $\pi^{-1}(\mathcal{U})$ = cover of E. Define

Ω^*_{cv} = presheaf over M

$$\Omega^*_{cv}(M) = \Omega^*_{cv}(\overbrace{\pi^{-1}(U)}^{E|_U}).$$

Then

$$0 \to \Omega^*_{cv}(E) \to C^0(\mathcal{U}, \Omega^*_{cv}) \to C^1(\mathcal{U}, \Omega^*_{cv}) \to \cdots$$

is exact (same proof as MV).

$$C^*(\mathcal{U}, \Omega^*_{cv}) \xrightarrow{E_1'} \begin{array}{|ll} \vdots & \\ \Omega^2_{cv}(E) & 0 \\ \Omega^1_{cv}(E) & 0 \\ \Omega^0_{cv}(E) & 0 \\ \hline \end{array} \xrightarrow{E_2'} \begin{array}{|ll} \vdots & \\ H^2_{cv}(E) & 0 \\ H^1_{cv}(E) & 0 \\ H^0_{cv}(E) & 0 \\ \hline \end{array} = E_\infty'.$$

$$\Big\downarrow E_1$$

$$C^*(\mathcal{U}, \mathcal{H}^*_{cv})$$

where $\mathcal{H}^*_{cv}$ = presheaf over M, $\mathcal{H}^*_{cv}(U) = H^*_{cv}(E|_U)$.

Poincaré Lemma: $H^k_{cv}(E|_U) = H^k_{cv}(U \times \mathbb{R}^n)$ integrate along fibers

$\cong H^{k-n}(U)$ (same pf as Poincaré lemma)

$\cong \begin{cases} \mathbb{R} & k=n \\ 0 & \text{otherwise.} \end{cases}$

$$E_1 = \begin{array}{c|ccc} & & 0 & \\ n & C^0(\mathcal{U}, \mathcal{H}^n_{cv}) & C^1(\mathcal{U}, \mathcal{H}^n_{cv}) & \cdots \\ & & 0 & \\ \hline \end{array} \xrightarrow{E_2} \begin{array}{|ccc} & 0 & \\ \check{H}^0(\mathcal{U}, \mathcal{H}^n_{cv}) & \check{H}^1(\mathcal{U}, \mathcal{H}^n_{cv}) & \cdots \\ & 0 & \\ \hline \end{array}$$

So: $\underline{H^*_{cv}(E) \cong \check{H}^{*-n}(\mathcal{U}, \mathcal{H}^n_{cv}).}$ (doesn't need orientability)

In the orientable case:

<u>Lemma</u> $_{\wedge}$ (E orientable.) $\exists$ global form $\Phi \in \Omega^n_{cv}(E)$, $d\Phi = 0$, such that for each fiber $E_x \cong \mathbb{R}^n$, $i^*([\Phi])$ generates $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$.

$$\left(\begin{array}{c} E_x \xhookrightarrow{i} E \\ H^n_{cv}(E) \xrightarrow{i^*} H^n_c(E_x) \end{array} \right)$$

<u>Then</u>: $\mathcal{H}^n_{cv}(U) \cong \mathbb{R}$ is generated by $[\Phi|_{\pi^{-1}(U)}]$
and for $V \subset U$, the restriction map $\mathcal{H}^n_{cv}(U) \to \mathcal{H}^n_{cv}(V)$ sends $\Phi|_{\pi^{-1}(U)}$ to $\Phi|_{\pi^{-1}(V)}$.

So $\mathcal{H}^n_{cv} \xrightarrow{\cong} \underline{\mathbb{R}}$ and thus $H^*_{cv}(E) \cong \check{H}^{*-n}(\mathcal{U}, \mathcal{H}^n_{cv})$

$$\mathcal{H}^n_{cv}(U) \longrightarrow \mathbb{R} \qquad \cong \check{H}^{*-n}(\mathcal{U}, \underline{\mathbb{R}})$$

$$\Phi|_U \longmapsto 1 \qquad \cong H^{*-n}(M).$$

$[\Phi] \in H^n_{cv}(E)$ is called the <u>Thom class</u> of E.
(think: volume bump form on each E_x)

Stronger form of Thom isomorphism

$\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ orientable rank n.

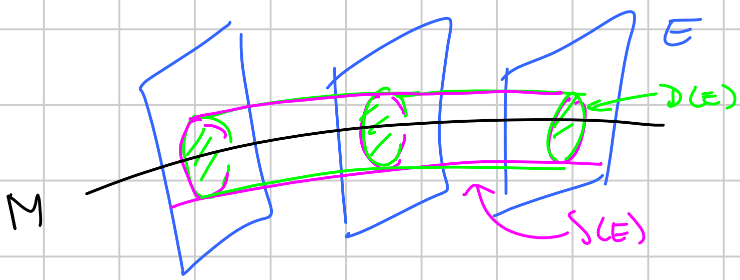
$\exists [\Phi] \in H^n_{cv}(E)$ with $[\Phi]|_{E_x} \in H^n_c(\mathbb{R}^n)$ having $\int = 1$ s.t.

$$\begin{aligned} H^*(M) &\longrightarrow H^{*+n}_{cv}(E) \\ [\omega] &\longmapsto [\pi^*\omega \wedge \Phi] \end{aligned}$$ is an isom.

Algebraic form of Thom (see Milnor-Stasheff)

$\mathbb{R}^n \to E$, $\downarrow$, M vector bundle with a metric $\langle , \rangle$ on E_x $\forall x$. Define the

- disk bundle $D(E) = \{v \in E \mid \|v\| \leq 1\}$
- sphere bundle $S(E) = \{v \in E \mid \|v\| = 1\}$



$$H^*_{cv}(E) \cong H^*(D(E), S(E); \mathbb{R})$$ relative cohomology.

$$\omega \longmapsto \left(\sigma \mapsto \int_\sigma \omega\right)$$

supp $\omega \subset D(E)$

Thm If $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is orientable, then $\exists\ \Phi \in H^n(D(E), S(E); \mathbb{Z})$ s.t. the map

$$\begin{aligned} H^*(M, \mathbb{Z}) &\longrightarrow H^{*+n}(D(E), S(E); \mathbb{Z}) \\ \alpha &\longmapsto \pi^*\alpha \cup \Phi \end{aligned}$$

is an isomorphism.

(If non-orientable: still true, with $\mathbb{Z}/2$ coeffs.)

Poincaré Duality

M^n = oriented mfd, not nec cpt.

N^{n-k} = (topologically) closed, oriented submfd, codim k, $N \overset{i}{\hookrightarrow} M$.

N induces a map

$$\varphi_N : \Omega_c^{n-k}(M) \to \mathbb{R} \qquad \rightsquigarrow \qquad \varphi_N : H_c^{n-k}(M) \to \mathbb{R} \text{ by Stokes,}$$
$$\omega \mapsto \int_N i^*\omega \qquad\qquad \varphi_N \in \left(H_c^{n-k}(M)\right)^*.$$

By Poincaré duality, $\left(H_c^{n-k}(M)\right)^* \cong H^k(M)$

so there's a class

$$PD(N) \in H^k(M) \qquad \left(\text{i.e. } \int_N i^*\omega = \int_M \omega \wedge PD(N)\right)$$

called the <u>Poincaré dual</u> of N.

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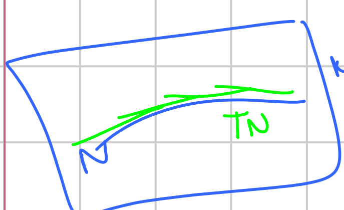
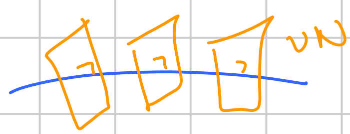
We can visualize this using Thom.

(M,g) Riem mfd, $N^{n-k} \subset M$. The <u>normal bundle</u> to N is the rank k vector bundle $\nu N \subset TM$, $\begin{array}{c}\mathbb{R}^k \to \nu N \\ \downarrow \\ N\end{array}$, defined by

$$(\nu N)_x = \{ v \in T_xM \mid v \perp T_xN \} = (T_xN)^\perp .$$





For any vector bundle $\begin{array}{c} E \\ \downarrow \\ M \end{array}$, the <u>zero section</u> is $s_0 : M \to E$, $s_0(x) = 0 \in E_x$.

<u>Tubular Neighborhood Thm</u> N has a neighborhood $\mathrm{nbd}(N) \subset M$ s.t.

$$\begin{array}{ccc} \nu N & \cong & \mathrm{nbd}(N) \\ \cup & & \cup \\ 0 \text{ section} & \longrightarrow & N \end{array}$$

Now: any cv form on νN is a form on M: use Tub. Nbd. Thm. + extend by 0 to $M - \text{nbd}(N)$.

Thus we get a map $\Omega^k_{cv}(\nu N) \xrightarrow{i_*} \Omega^k(M) \rightsquigarrow H^k_{cv}(\nu N) \xrightarrow{i_*} H^k(M)$.

Now νN has Thom class $[\Phi] \in H^k_{cv}(\nu N)$:

<u>Thm</u> (p.67) $\quad i_*[\Phi] = PD(N) \in H^k(M)$.

We can interpret $\cup$ (cup product) on $H^*(M)$ in terms of Poincaré duals.

$N_1^{n-k_1}, N_2^{n-k_2} \subset M \rightsquigarrow PD(N_1) \in H^{k_1}(M),\ PD(N_2) \in H^{k_2}(M)$.

What's $PD(N_1) \cup PD(N_2) \in H^{k_1+k_2}(M)$?

Suppose N_1, N_2 intersect <u>transversely</u>: $\forall x \in N_1 \cap N_2$,

$\text{codim}(T_xN_1 \cap T_xN_2) = \text{codim}\, T_xN_1 + \text{codim}\, T_xN_2$.

Then $N_1 \cap N_2$ is a smooth $(n-k_1-k_2)$-diml submfd of M.

$$\nu(N_1 \cap N_2) = \nu(N_1) \oplus \nu(N_2)$$

(Property of Φ p. 65) $\Rightarrow$ $\Phi(\nu(N_1 \cap N_2)) = \Phi(\nu(N_1) \oplus \nu(N_2)) = \Phi(\nu(N_1)) \cup \Phi(\nu(N_2))$

$$\Rightarrow \boxed{PD(N_1 \cap N_2) = PD(N_1) \cup PD(N_2)} \quad \text{(p. 69)}.$$

Cup product on cohom $\xleftrightarrow{PD}$ intersection of submfds.

Special case: if $k_1+k_2=n$:

$$\underset{H^k}{\underset{\cap}{PD(N_1)}}\,\underset{H_{n-k}}{\underset{\cap}{[N_2]}} = \int_{N_2} PD(N_1) = \int_M PD(N_1)\cup PD(N_2) = \int_M PD(N_1\cap N_2)$$

$$= \int_{N_1\cap N_2} 1 = \#(N_1\cap N_2).$$

Euler class

$\mathbb{R}^k \to E$
$\quad\downarrow$
$\quad M$ oriented v.b., $s_0: M\to E$ zero section, $s_0^*: H^*_{cv}(E)\to H^*(M)$.

Def The Euler class of E is

$$e(E) = s_0^*[\Phi] \in H^k(M).$$

This is an example of a characteristic class: invt of vector bundle up to $\cong$.
(oriented $\rightsquigarrow e(E)$; complex $\rightsquigarrow c_k(E)$; real $\rightsquigarrow$ Pontryagin).

Facts:

- If E = trivial $= M\times\mathbb{R}^n$ then $e(E)=0$
- $e(E)$ is PD to the zero locus of any section $s: M\to E$ (p.134)
- if $S(E)$ = sphere bundle to E then $e(E)$ is what shows up in the Gysin sequence
- if $E=TM$ and M^n is compact then $e(TM)\in H^n(M)$:

$$\int_M e(TM) = \chi(M) \qquad \text{Euler characteristic!}$$

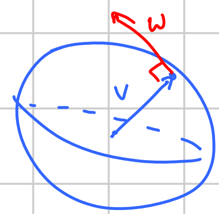
$\underline{\text{Ex}}$ $M = S^2$, $\quad e(TS^2) = 2 \in H^2(S^2; \mathbb{Z})$

Sphere bundle
$$\begin{array}{c} S^1 \to STS^2 \\ \qquad\ \downarrow \\ \qquad\ S^2 \end{array}$$

Gysin sequence
(works over either $\mathbb{Z}$ or $\mathbb{R}$)

$$\begin{array}{ccccc}
 & & & \hookrightarrow 0 & \\
0 & \longrightarrow & H^3(STS^2) & \longrightarrow & H^2(S^2) \\
\xrightarrow{\times 2} H^2(S^2) & \longrightarrow & H^2(STS^2) & \longrightarrow & \cancel{H^1(S^2)}\ 0 \\
0\ \cancel{H^1(S^2)} & \longrightarrow & H^1(STS^2) & \longrightarrow & H^0(S^2) \\
\hookrightarrow H^0(S^2) & \longrightarrow & H^0(STS^2) & \longrightarrow & 0 \\
 & & & & 0
\end{array}$$

$\Rightarrow$ $H^0(STS^2) = \begin{cases} \mathbb{R} \\ \mathbb{Z} \end{cases}$ $\quad H^1(STS^2) = 0 \quad$ $H^2(STS^2) = \begin{cases} 0 \\ \mathbb{Z}/2 \end{cases}$ $\quad H^3(STS^2) = \begin{cases} \mathbb{R} \\ \mathbb{Z} \end{cases}$.

In fact, point in STS^2 $\longleftrightarrow$  $\perp$ unit vectors v, w

$\longleftrightarrow$ element of $SO(3)$

so $STS^2 \cong SO(3) \cong \mathbb{RP}^3$.

Poincaré-Hopf

Let V = vector field on M: $V \in \Gamma(TM)$.
Each zero of V has an index: let $x \in M$, $V(x)=0$, assume x = isolated zero (true for generic). TM is locally trivial: if we chose a metric on M, then

$$TM|_{D^n_\epsilon(x)} \cong D^n_\epsilon(x) \times \mathbb{R}^n.$$

Restrict V to $S^{n-1}_\epsilon(x)$:

$$S^{n-1} = S^{n-1}_\epsilon(x) \xrightarrow{V} \mathbb{R}^n \setminus 0 \longrightarrow S^{n-1}$$

and define

$$\text{index}(x) := \text{degree}(S^{n-1} \longrightarrow S^{n-1}).$$

Poincaré-Hopf thm (p. 129)

$$\chi(M) = \sum_{x = \text{zeros of } V} \text{index}(x).$$